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POST-CRITICAL PLASTIC DEFORMATION OF BIAXIALLY STRETCHED SHEETS

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Abstract—A theoretical and numerical analysis of the formation of a localized neck in a biaxially stretched sheet is presented. A time-independent constitutive law is assumed to be incrementally non-linear as suggested by micromechanical studies of the elastoplastic deformation of polycrystalline metals. The incipient width of a necking band in an infinitely thin perfect sheet of a time-independent material is found here to have a well-defined initial value, proportional to the in-plane sheet dimension. During subsequent post-critical deformation the boundary of the necking band moves with respect to the material until the transition to localized necking is completed. These conclusions are derived on a theoretical route from the condition of stability of the post-bifurcation deformation process and are confirmed by the numerical analysis performed for a sheet of finite thickness.

1. INTRODUCTION

According to Hills' (1952) theory, localized necking in thin sheets in the plane stress idealization corresponds to a discontinuity of velocity, or alternatively to a vanishingly thin band where the strain rate is infinite relative to that outside the band. In biaxially stretched sheets the localized necking can be preceded by a transitory (quasi-stable) process of concentration of deformation [cf. Marciniak (1978)]. It is the purpose of this paper to provide a theoretical and numerical analysis of that process in a sheet of finite in-plane dimensions, when in-plane diffuse necking is disregarded or excluded by kinematic boundary conditions. A novel feature of the present study is that the width of the zone where the strain is concentrating can be determined and can decrease gradually in time.

For the plane stress time-independent model of a perfect sheet and in the circumstances to be specified later, the transitory process starts at the bifurcation point when the linearized equations of continuing equilibrium loose ellipticity. The time-independent model with no intrinsic length variable has been regarded as providing an undetermined or vanishing width of the subsequently forming localization band. Contrary to that widespread opinion, it will be shown below that the incipient width of the necking band (or bands) can be calculated, being proportional to the in-plane dimension of the sheet. During further stretching, the boundary of the necking band need not be fixed but can propagate with respect to the material until the transition to localized necking is completed.

There are two essential ingredients of the theory used to obtain those results. First, the time-independent incremental constitutive law for an elastic–plastic material is not restricted to have only two linear branches corresponding either to loading or to unloading as in the classical elastoplasticity, but is allowed to admit an arbitrarily non-linear velocity-gradient potential. This is in accord with micromechanical studies of elastoplastic behaviour of polycrystalline metals (Hill, 1967; Hutchinson, 1970) which predict the existence of a non-linear transitory range in the stress- or strain-rate space between the constitutive cones of "total" loading and elastic unloading, associated with formation of a vertex on the yield surface. If the classical elastoplastic model with a smooth yield surface is used then unrealistically high values of the strains at the onset of localized necking under kinematically controlled biaxial stretching are obtained, unless sufficiently strong initial imperfections are introduced as in the MK-approach (Marciniak and Kuczyński, 1967). On the other hand, Stören and Rice (1975) have demonstrated that the use of the total loading moduli at a yield surface vertex can reduce the predicted critical strains at the onset of localization to the experimentally observed level without the need of appealing to imperfections.† To study subsequent development of a necking band, the non-linear range of the incremental constitutive law has to be specified. Christoffersen and Hutchinson (1979) have applied their phenomenological model of the elastoplastic response at a yield surface vertex, known as the J_2 corner theory of plasticity. They used the MK-approach and assumed that the straining history outside the band is uniform and given independently of the necking process.

Here, the interaction between the deformation inside and outside a necking band is not disregarded. Under the assumptions of plane stress and after reducing the problem to one dimension, the bifurcation theory predicts an inherent indeterminacy in the localization process. As discussed in Section 3, infinitely many incremental solutions associated with different band width evolution can exist at every stage of the post-critical deformation, including the critical instant, all the solutions being correct from a mathematical point of view. However, they need not to be equally correct physically. The actual deformation process must in some sense be *stable* in order to be realizable in a physical system. As the second essential ingredient of the theory used here, we shall apply the energy criterion of instability of a deformation *process*, formulated by Petryk (1985) under the general assumptions of Hill's (1959, 1978) theory of bifurcation: justification of the criterion has been given by Petryk (1991, 1992). Petryk and Thermann (1992) have shown how the criterion can be used in numerical calculations to select the possibly stable post-bifurcation branch: the narrower concept of instability of *equilibrium* is known to be insufficient for this purpose. In the present paper the criterion is used to generate an additional equation which removes the aforementioned indeterminacy of post-critical deformation.

In a perfect sheet of finite thickness, the necking can start without loss of ellipticity of the governing (linearized) equations. Contrary to the plane stress case, the post-bifurcation process of strain concentration can be determined numerically in a straightforward manner so long as local (surface or material) instabilities do not come into play. In Section 4 a comparison is made between the results of finite element calculations for a finite sheet thickness and the plane stress results based on the stability condition. As a necessary preliminary step to a future analysis of more complex boundary conditions, the idealized problem with the deformations being uniform in the direction of the smaller principal stretch has been considered. The material has been assumed to obey the finite strain version of the J_2 corner theory of plasticity so that a comparison can also be made with the previous calculations done by Christoffersen and Hutchinson (1979).

2. FORMULATION OF THE PROBLEM

Consider a plane sheet in the rectangular Cartesian coordinates (x_1, x_2, x_3) such that the x_3-axis is orthogonal to the sheet plane (Fig. 1). The sheet, initially homogeneous and isotropic in the plane (x_1, x_2), is subject to quasi-static biaxial stretching in the x_1 and x_2 directions. We assume that normal velocities and zero shear tractions are prescribed over the sheet end planes, $x_1 = -l$, $x_1 = l$ and $x_2 = 0$, $x_2 = l_2$, say, so that the averaged values $\bar{D}_1$, $\bar{D}_2$ of Eulerian strain rates D_{11}, D_{22} are given. Since the material is time-independent, only their ratio $\rho = \bar{D}_2/\bar{D}_1$ is relevant: we assume that $0 < \rho \leqslant 1$. The sheet surface is traction free. The ratio of a (non-uniform) current sheet thickness h to l is assumed to be small (infinitesimal in Section 3 but finite in Section 4); the dimension l_2 will have no significance.

† Recently, Hill (1991) suggested that anisotropic hardening may be the key to understanding localized necking without invoking yield vertices or Marciniak grooves, at least in certain circumstances. It is not evident, however, how to apply this idea to the problem of kinematically controlled stretching examined here.

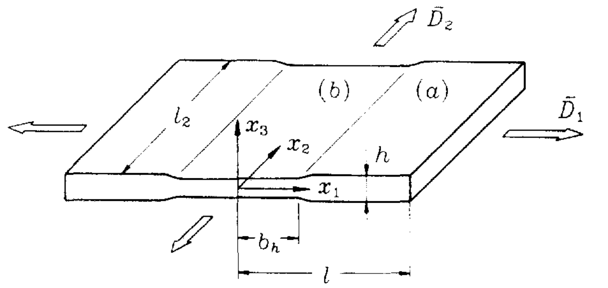


Fig. 1. A stretched sheet in the current configuration.

In Appendix A it is shown that so long as the linearized rate equations of continuing in-plane equilibrium under plane-stress conditions are strongly elliptic and of constant coefficients then they admit only one uniform solution to the boundary value problem specified above. The primary bifurcation mode $\Delta\mathbf{v}$ at the instant of ellipticity loss does not involve variations along the x_2-axis, at least for the coefficients determined from the J_2 deformation theory of plasticity. Accordingly, the analysis of non-uniform deformations shall be restricted here to a generalized plane problem in the (x_1, x_3) plane, all quantities of interest being independent of x_2 (Fig. 1); this is also in accord with the common experience that the localized neck forms normal to the direction of the larger stretch. The tangential components F_{21}, F_{23} and σ_{21}, σ_{23} of the deformation gradient $\mathbf{F}$ and Cauchy stress $\boldsymbol{\sigma}$, respectively, are assumed to vanish identically, and D_{22} always coincides with given $\bar{D}_2$.

Constitutive rate equations for a time-independent material undergoing isothermal deformation can be written as follows (with the summation convention adopted for repeated subscripts):

$$\overset{\nabla}{\tau}_{ij} = \overset{\nabla}{\tau}_{ij}(\mathbf{D}, \mathscr{H}) = L_{ijkl}(\mathbf{D}, \mathscr{H})D_{kl}, \quad L_{ijkl} \equiv \partial\overset{\nabla}{\tau}_{ij}/\partial D_{kl}, \tag{1}$$

where $\overset{\nabla}{\tau}_{ij}$ are components of the corotational (Zaremba–Jaumann) flux of the Kirchhoff stress $\boldsymbol{\tau} = \det(\mathbf{F})\boldsymbol{\sigma}$, and the symbolic parameter $\mathscr{H}$ denotes the current state of the material, dependent on the prior deformation history. Arbitrary "hardening" or "softening" characteristics are allowed. Taking for simplicity the current configuration as the reference, it will be convenient to rewrite eqn (1) as

$$\dot{S}_{ij} = \dot{S}_{ij}(\dot{\mathbf{F}}, \mathscr{H}) = C_{ijkl}(\mathbf{D}, \mathscr{H})\dot{F}_{kl}, \quad C_{ijkl} \equiv \partial\dot{S}_{ij}/\partial\dot{F}_{kl}, \tag{2}$$

$$C_{ijkl} = L_{ijkl} - \tfrac{1}{2}(\sigma_{ik}\delta_{jl} + \sigma_{jk}\delta_{il} + \sigma_{il}\delta_{jk} - \sigma_{jl}\delta_{ik}), \quad \dot{F}_{ij} = D_{ij} + \Omega_{ij}. \tag{3}$$

Here, S_{ij} are components of the first Piola–Kirchhoff stress tensor (i.e. of the transposed nominal stress tensor), $\dot{\mathbf{F}}$ coincides with the velocity gradient, a dot over a symbol denotes the forward rate, δ_{ij} is the Kronecker delta and $\boldsymbol{\Omega}$ is the material spin.

We shall assume, following Hill (1959, 1978), that the constitutive rate equations (2) admit a velocity-gradient potential (continuously differentiable and homogeneous of degree two in $\dot{\mathbf{F}}$), viz.

$$\dot{S}_{ij} = \frac{\partial U}{\partial \dot{F}_{ij}}, \quad U = \tfrac{1}{2}\dot{S}_{ij}\dot{F}_{ij} \Leftrightarrow \overset{\nabla}{\tau}_{ij} = \frac{\partial \omega}{\partial D_{ij}}, \quad \omega = \tfrac{1}{2}\overset{\nabla}{\tau}_{ij}D_{ij}, \tag{4}$$

so that

$$C_{ijkl} = C_{klij}, \quad L_{ijkl} = L_{klij}. \tag{5}$$

The material "stiffness" moduli C_{ijkl}, L_{ijkl} can depend on the *direction* of $\mathbf{D}$ in a non-linear and piecewise continuous manner, and are obviously undefined for $\mathbf{D} = \mathbf{0}$ unless U is quadratic in $\dot{\mathbf{F}}$. The equations of classical plasticity with a smooth yield surface and the normality flow rule represent the special case of eqn (4) with L_{ijkl} being constant on each side of a hyperplane in **D**-space.

3. PLANE STRESS IDEALIZATION

Throughout this section we make use of the additional simplifying assumption of the plane stress state being uniform across the sheet thickness $h(x_1) \ll l$. We assume that $\dot{F}_{ij}$ and S_{ij} for $i \neq j$ can be taken as zero identically, and work with the principal values D_k and S_k. The strain rate D_3 at given $\mathscr{H}$ is assumed to be a single-valued function of (D_1, D_2), determined from the condition $\dot{S}_3 = 0$ (provided $C_{3333} \neq 0$). The plane stress specialization of the constitutive potential U is defined by

$$\hat{U}(D_1, D_2, \mathscr{H}) = U(D_1, D_2, D_3(D_1, D_2, \mathscr{H}), \mathscr{H}), \tag{6}$$

with constitutive rate equations

$$\dot{S}_i = \partial\hat{U}/\partial D_i = \hat{C}_{ik}(D_1, D_2, \mathscr{H})D_k \tag{7}$$

and plane stress moduli

$$\hat{C}_{ik} = \frac{\partial^2\hat{U}}{\partial D_i \partial D_k} = C_{ik} - C_{i3}C_{3k}/C_{33}, \quad i, k = 1, 2, \quad C_{ik} \equiv C_{iikk}(\text{no sum}). \tag{8}$$

All quantities of interest are now independent of x_2 and x_3. At a given stage of non-uniform deformation of the sheet, $\mathscr{H}$ may formally be replaced by x_1; in the formulae below the obvious dependence on x_1 will not be indicated explicitly to simplify the notation. Without loss of generality we can assume that the deformations are symmetric with respect to the point $x_1 = 0$ where the velocity $v_1(0) = 0$. A quasi-static solution to the rate boundary value problem is represented by a velocity function $v_1 : [0, l] \to \mathbf{R}$, assumed to be continuous and piecewise continuously differentiable. The set of (kinematically admissible) functions v_1 which satisfy the kinematical boundary conditions:

$$v_1(0) = 0, \quad v_1(l) = l\bar{D}_1 \tag{9}$$

will be denoted by $\mathscr{V}_1$. A velocity solution is a function $v_1 \in \mathscr{V}_1$ such that on substituting $D_1 = v_1' \equiv \mathrm{d}v_1/\mathrm{d}x_1$ and $D_2 = \bar{D}_2$ into eqn (7) it satisfies the condition of continuing equilibrium

$$(h\sigma_1)^{\boldsymbol{\cdot}} = \text{const} \Leftrightarrow h\dot{S}_1 = \text{const} \quad \text{for } x_1 \in (0, l); \tag{10}$$

it is recalled that the current configuration is taken as reference.

3.1. *Stability condition*

According to the energy criterion of plastic instability (Petryk, 1985, 1991) specified for the present problem with zero potential energy of external loads, a solution v_1^0 can correspond to a stable deformation path only if it minimizes the second-order work functional among all kinematically admissible velocity functions, viz.

$$\int_0^l \hat{U}(v_1', \bar{D}_2)h\,\mathrm{d}x_1 \geqslant \int_0^l \hat{U}(v_1^{0\prime}, \bar{D}_2)h\,\mathrm{d}x_1 \quad \text{for every } v_1 \in \mathscr{V}_1. \tag{11}$$

On the other hand, any solution v_1, stable or not, assigns to the left-hand functional a stationary value in $\mathscr{V}_1$ [cf. Hill (1959)].

The integral condition (11) is easily shown to be equivalent to the following pointwise condition:

$$\hat{\mathscr{E}}(D_1^0, D_1; \bar{D}_2) \geqslant 0 \quad \text{for every } D_1 \quad \text{and at any } x_1 \in (0, l) \tag{12}$$

provided x_1 is a point of differentiability of v_1^0. Equation (12) is the classical Weierstrass condition of the calculus of variations for the minimization problem (11). The expression

$$\hat{\mathscr{E}}(D_1^0, D_1; \bar{D}_2) = \hat{U}(D_1, \bar{D}_2) - \hat{U}(D_1^0, \bar{D}_2) - \dot{S}_1^0 \cdot (D_1 - D_1^0) \tag{13}$$

defines the Weierstrass function associated with $\hat{U}(\cdot, \bar{D}_2, \mathscr{H})$, representing the excess of $\hat{U}$ at D_1 over its linear approximation constructed at D_1^0. Here and in the following, the quantities distinguished by the superscript zero, are associated with v_1^0.

In Appendix A the circumstances are specified in which along a uniform deformation path the uniqueness of a velocity solution and the incremental stability of deformation in the sense of eqn (11) are lost exactly at the point of ellipticity loss determined for the *tangent* moduli, that is, when a material state $\mathscr{H}^{\text{crit}}$ is reached such that the *linearized* bifurcation condition

$$\hat{C}_{11}(\bar{D}_1, \bar{D}_2, \mathscr{H}^{\text{crit}}) = 0 \tag{14}$$

is met; this is the critical point examined by Stören and Rice (1975) in the case of biaxial stretching.

The post-critical solution for the plane stress problem is not determined by the kinematic and static conditions alone. To select among infinitely many mathematically admissible alternatives the solution of physical meaning, we shall require the stability condition (12) to hold along the deformation path. This is the essence of the new approach developed here. Some details of the analysis are given in Appendix B; more essential implications are listed below.

3.2. *The incipient width of necking band*

Assume for simplicity that $\bar{\mathbf{D}}$ at $\mathscr{H}^{\text{crit}}$ corresponds to continuing proportional straining and represents the axis of the constitutive cone of total loading within which the moduli are independent of $\mathbf{D}$. The total loading cone is defined by $\phi < \theta_0$, where θ_0 is the cone angle and $\phi \geqslant 0$ is an angle (in general, defined in terms of a scalar product generated by an arbitrary positive-definite fourth-order tensor) between $\mathbf{D}$ and $\bar{\mathbf{D}}$.

Denote by $\bar{\phi}$ the angle between the plane strain rate $(\bar{D}_k) = (1, 0, D_3(1, 0))$ and $(\bar{D}_k)$. It is found that if $\bar{\phi} < \theta_0$ at $\mathscr{H}^{\text{crit}}$ then, as a rule, any quasi-static continuation of the solution (for a vanishingly thin sheet) violates the requirement of stability of equilibrium and is thus unlikely to have a physical meaning.

Suppose thus that in the uniform critical state $\mathscr{H}^{\text{crit}}$ satisfying eqn (14) we have $\bar{\phi} > \theta_0$ and, moreover, $\hat{C}_{11} > 0$ for $\phi > \theta_0$. The graph of $\hat{U}$ as a function of D_1 at given $D_2 = \bar{D}_2$ in this state takes the form visualized in Fig. 2(b), with the linear range corresponding to total loading [Fig. 2(a)]. Infinitely many solutions in velocities can be constructed at $\mathscr{H}^{\text{crit}}$. In particular, an arbitrary pair $(D_1^{(b)}, D_1^{(a)})$ taken from the linear range of $\hat{U}$ (including the limits D_1^- and D_1^+ at which $\phi = \theta_0$), such that $D_1^{(b)} > \bar{D}_1 > D_1^{(a)}$, defines a piecewise uniform solution $v_1 \in \mathscr{V}$ which describes incipient necking in a single band of width b. Such solutions are obtained by assuming $v_1'(x_1) = D_1^{(b)}$ for $0 < x_1 < b$ and $v_1'(x_1) = D_1^{(a)}$ for $b < x_1 < l$, with

$$b = \frac{\bar{D}_1 - D_1^{(a)}}{D_1^{(b)} - D_1^{(a)}}\, l \tag{15}$$

in order to satisfy eqn (9). The static condition (10) is evidently satisfied.

Under the assumptions specified, all the solutions satisfy eqn (12) *at* the critical state $\mathscr{H}^{\text{crit}}$. In the typical circumstances indicated in Appendix B, the solution v_1^0 which satisfies the condition (12) just *beyond* $\mathscr{H}^{\text{crit}}$ (if it exists) must additionally satisfy $\phi = \theta_0$ at $\mathscr{H}^{\text{crit}}$ at every point of differentiability of v_1^0, i.e. not only within the incipient necking band but also

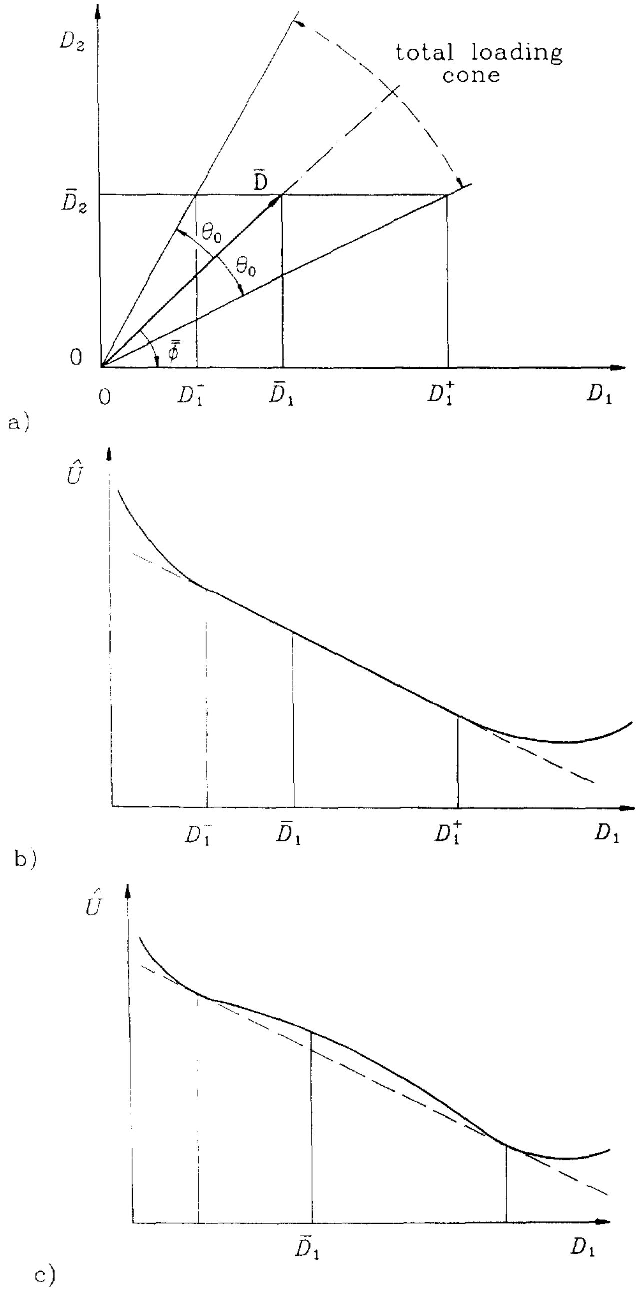


Fig. 2. (a) Strain rates and angles associated with the constitutive cone of total loading. (b) Schematic graph of the plane stress constitutive potential $\hat{U}$ vs D_1 for fixed $\bar{D}_2$ at the critical stage [eqn (14)]. (c) The graph beyond the critical stage.

outside the band, so that $D_1^{(b)} = D_1^-$, $D_1^{(a)} = D_1^+$. On determining these values from trigonometric relationships obtained with the help of Fig. 2(a), substituting into eqn (15) and rearranging, we arrive at the following compact formula:

$$h = \frac{1}{2}\left(1 - \frac{\tan\theta_0}{\tan\bar{\phi}}\right) l \quad \text{if } \bar{\phi} > \theta_0 \tag{16}$$

for the incipient width of the necking band in a perfect infinitely thin sheet of finite in-plane dimension l. The assumption about formation of just one band under the adopted boundary conditions is supported by the analysis of neck formation in a sheet of finite thickness; cf. Section 4. If this assumption is dropped then the number and particular widths of necking bands become undefined but their *total* incipient width can still be determined from eqn (16).

3.3. *Post-critical behaviour*

Without loss of generality we take the prescribed overall logarithmic strain in the x_1 direction, $\bar{\varepsilon}_1 = \int \mathrm{d}l/l$, as a time measure identified with t, so that $\bar{D}_1 = 1$. The analysis shows that two zones (b) and (a) of uniform straining can still exist in the post-critical range but being separated from each other† by an evolving transitory zone of non-uniform straining. A peculiar feature of the present problem is that along *any* post-critical path (stable or not) involving a homogeneous zone where the graph of $\hat{U}$ vs D_1 has the qualitative form shown in Fig. 2(c) or (b), there exist infinitely many velocity solutions at *every* instant. The indeterminacy is again removed with the help of the stability condition (12). In the examples examined in this paper, the post-critical $\hat{U}$-graph in zone (b) is non-convex as in Fig. 2(c), and the solution path corresponding to a necking band of a fixed nominal width (i.e. when the band boundary is not moving with respect to the material) has been found to be unstable. A solution corresponding to the nominal width of zone (b) decreasing in time has been searched for. At the moving end-point $x_1^{(b)}$ of zone (b) $(0 < x_1^{(b)} \leqslant b)$ the strain rate (but not h) suffers a jump from the value $D_1^{(b)}$ in zone (b) to $D_1^{(c)}$, say, such that

$$\dot{S}\,(D_1^{(b)}, \bar{D}_2, \mathscr{H}^{(b)}) = \dot{S}\,(D_1^{(c)}, \bar{D}_2, \mathscr{H}^{(b)}). \tag{17}$$

To satisfy eqn (12) at both sides of the discontinuity point, we must also have

$$\hat{\mathcal{E}}\,(D_1^{(b)}, D_1^{(c)}; \bar{D}_2) = 0 \quad \text{at } \mathscr{H}^{(b)} \tag{18}$$

at any instant; this is the additional equation derived from the path stability requirement [eqn (11)].

The two unknowns: $D_1^{(b)}(t)$, $D_1^{(c)}(t)$, and thus the deformation history in zone (b), can now be determined from eqns (17) and (18) without any reference to deformation in other parts of the sheet. Once this has been done, the function $D_1^{(a)}(t)$ in zone (a) can be determined from eqn (10) and then $b(t)$ can be found from eqn (15). The transition to localized necking $(D_1^{(b)} \to +\infty$ and $b \to 0)$ is accompanied by approaching the limit of stability of equilibrium.

If a small imperfection is introduced as in the MK-approach, then the above analysis still applies to the initially homogeneous part of the sheet outside the weakened zone. The localized necking in the weakened zone can occur before b calculated as above falls to zero but the difference should tend to zero with the vanishing magnitude of imperfection; this is confirmed below in examples.

3.4. *Examples*

The equations governing the strain development in zones (b) and (a) have been solved numerically by employing an incremental procedure. A finite strain version of the J_2 corner theory (Christoffersen and Hutchinson, 1979) has been used as an approximation‡ of the incrementally non-linear plastic response of the material at the current vertex point on the yield surface. Immediately beyond the critical point the graph of $\hat{U}$ vs D_1 becomes non-convex, and to satisfy eqn (12) the strain rates $\mathbf{D}^{(a)}$ and $\mathbf{D}^{(b)}$ must leave the total loading cone where the moduli are assumed to be those of the deformation theory of plasticity. In the calculations performed, the solution with a fixed nominal width of the necking band violated condition (12) in zone (b) while the solution determined with the help of eqns (17) and (18) was found to be acceptable.

† In fact, the order of appearance of different zones along the x_1-axis is not essential here. The order has been chosen to represent a solution with one necking band, obtained in the limit as a finite sheet thickness tends to zero.

‡ As indicated in Appendix A, this approximation is not fully satisfactory.

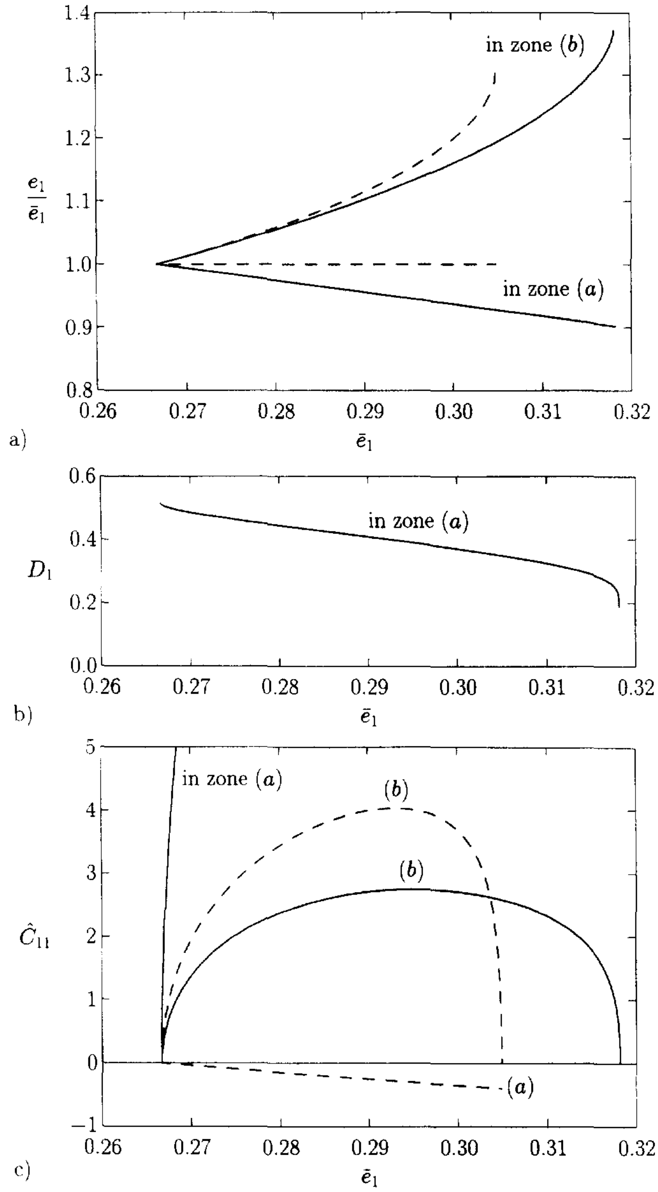


Fig. 3. Post-critical evolution of (a) strain ratio $e_1/\bar{e}_1$, (b) strain rate $D_1^{(a)}$, (c) plane stress modulus $\hat{C}_{11}$, with overall logarithmic strain $\bar{e}_1$, in zone (b) of concentrating deformation and in zone (a) outside the necking band, for $\rho = 1$. Solid lines correspond to the band width determined from the energy condition of path stability, and dashed lines to a vanishingly narrow necking band. In all cases (and for all next figures), $N = 0.2$, $\theta_0 = 22.5^\circ$, $\theta_c = 135.0^\circ$.

Sample results of the calculations are visualized in Figs 3 and 4. The material parameters and the strain ratio $\rho = 1$ are taken as in one of the examples from Christoffersen and Hutchinson (1979) in order to highlight certain similarities, and also significant differences between their results and the present ones. We take an exponent of the power hardening law $N = 0.2$, an angle of the constitutive cone of total loading $\theta_0 = 22.5^\circ$ and of elastic unloading $\theta_c = 135^\circ$; for other details see the paper just cited. The elastic moduli are taken to be sufficiently large (Young's modulus-to-stress ratio > 1000) to give practically the incompressible rigid-plastic behaviour assumed in the reference. The plots in Fig. 3(a) show the post-critical strain history in zones (b) and (a) compared with that determined as in Christoffersen and Hutchinson (1979), but with no imperfection (not computed in the reference); the latter case corresponds to the solution with $b = 0$ which we reject since it violates eqn (12). Figure 3(b) shows the significant difference between the strain rate outside the necking band and the prescribed averaged value $\bar{D}_1 = 1$. The plots of $\hat{C}_{11}^{(b)}$ and $\hat{C}_{11}^{(a)}$ (with

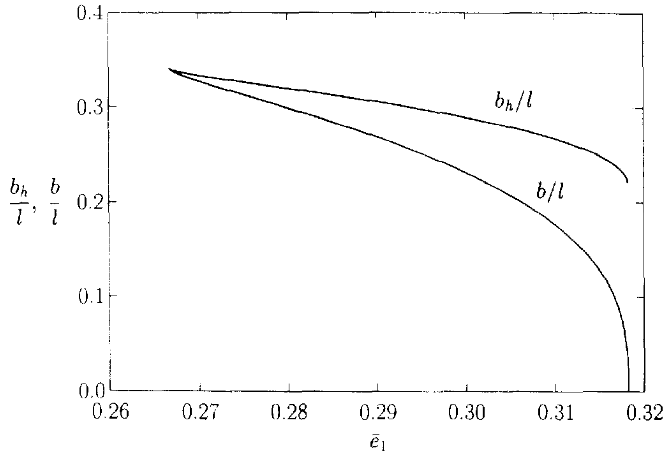


Fig. 4. Relative width of the current necking band, b/l, determined from strain-rate distribution, and of the neck, b_h/l, determined from thickness distribution, as a function of overall logarithmic strain $\bar{\varepsilon}_1$ (for $\rho = 1$).

the yield stress as unity) vs $\bar{\varepsilon}_1$ are presented in Fig. 3(c). The end points on all curves in Fig. 3, except on that for $\hat{C}_{11}^{(a)}$ which goes beyond the scale of the figure, correspond to the onset of localized necking in the plane stress idealization.

Figure 4 shows how the relative widths of the necking band, b/l, and of the neck, b_h/l, vary with the overall logarithmic strain $\bar{\varepsilon}_1$, the former (but not the latter) approaching zero at the limit of localized necking. The current width of the necking band, b, which corresponds to the non-uniformity of strain rate can still be conveniently defined by the formula (15), the superscripts (a) and (b) referring to quantities within zones (a) and (b), respectively. It differs, of course, from the current neck width b_h corresponding to the distribution of sheet thickness, defined by the analogous formula

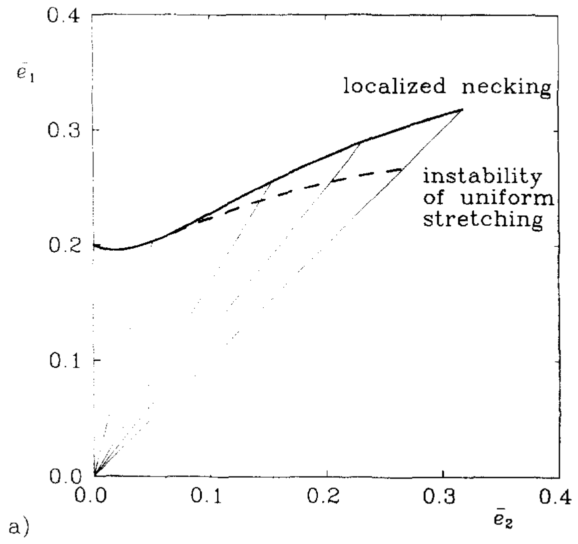
$$b_h = \frac{\bar{h} - h^{(a)}}{h^{(b)} - h^{(a)}}\, l, \tag{19}$$

where $\bar{h}$ is the averaged sheet thickness which can be approximately determined, e.g. by using the incompressibility condition. The incipient width calculated numerically† practically coincides with that found from eqn (16) with $\bar{\bar{\phi}}$ determined from eqn (B4) (the relative difference is less than 0.03%) on account of negligible elastic compliancies.

For the same material parameters as above, the forming limit diagram for proportional increasing of the overall logarithmic strains $\bar{\varepsilon}_1$, $\bar{\varepsilon}_2$ has been calculated as shown in Fig. 5(a). The dashed line represents the points of ellipticity loss and coincides with that determined by Stören and Rice (1975); however, it corresponds here to the onset of instability of uniform straining and not necessarily to the onset of localized necking. The onset of localized necking predicted by the present theory is marked by the solid line and occurs at a later stage of the non-uniform deformation provided ρ is not too small, in this computational case if $\rho > 0.24$. Only for smaller values of ρ does the localized necking occur as soon as the point of ellipticity loss is reached, since then $\bar{\bar{\phi}} < \theta_0$.

Still for the same data, two alternative forming limit diagrams have been determined [solid lines in Fig. 5(b)] which correspond to the limit *local* logarithmic strains in zones (a) or (b) at the onset of localized necking. Between these two lines, there is a whole family of forming limit curves which are dependent on a point in the transitory zone where the limit strain is determined. For a major part of the sheet (note that $b \leqslant l/2$) the lower line (a) is relevant. After crossing the Stören–Rice curve plotted as a dashed line, the local strain histories for a given ρ deviate from proportional straining, as depicted in the figure. Hence,

† More precisely, that found by backward extrapolation of the post-critical values, but the difference is immaterial.



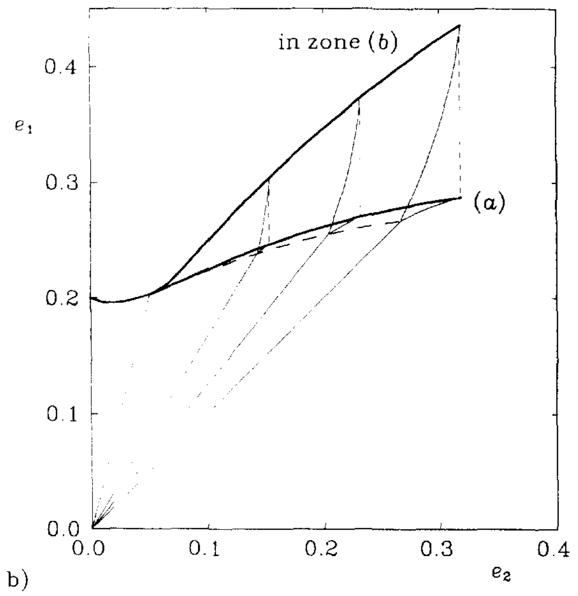


Fig. 5. Predicted forming limit diagram (a thick solid line): (a) in proportionally increasing overall strains $\bar{e}_2$, $\bar{e}_1$; (b) in local strains e_2, e_1 in zones (*b*) and (*a*). Beyond the Stören–Rice (1975) curve plotted as a dashed line, local strains become non-uniform and increase non-proportionally. Thin solid lines in Fig. 5(b) show particular strain paths for zones (*b*) or (*a*) up to the onset of localized necking, for $\rho = 1$, 0.8 or 0.6.

the theory predicts that the limit strain outside a final band of localized necking in an initially perfect sheet subject to proportional overall stretching is in general place-dependent and is reached on a non-proportional route.

This general conclusion remains valid for a sheet with a local imperfection provided the imperfection is sufficiently small. To show this, let us imagine, following Marciniak, that a very narrow band is initially slightly weaker (e.g. thinner) than the remaining homogeneous part of the sheet. If this band is vanishingly narrow then it does not contribute to the total extension of the sheet. This means that the above calculations of the stress and deformation history in zones (*a*) and (*b*) apply without any changes to the imperfect sheet outside the weakened band. The calculated stress history outside this band provides the static condition for S_1 which, with the kinematical condition for D_2, enable us to determine the deformation history at the weakest point of the whole sheet. The localized necking occurs (i.e. the strain rate at that point tends to infinity) when the plane stress modulus $\hat{C}_{11}$

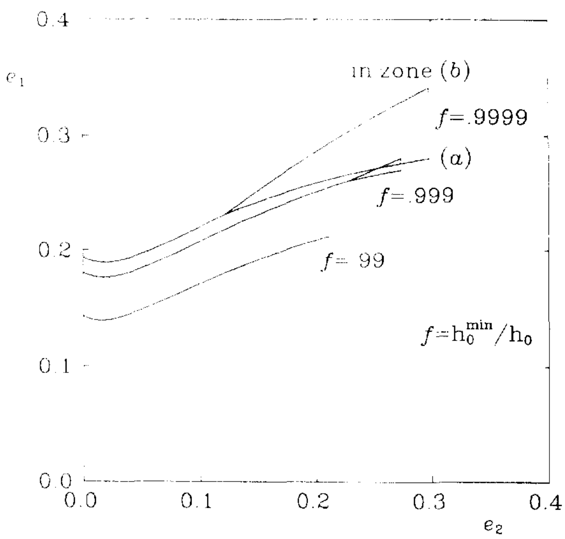


Fig. 6. Modification of the forming limit diagram from Fig. 5(b) for an imperfect sheet.

at that point falls to zero: this happens at a positive value of the band width b found from eqn (15).

The quantitative effect of the presence of the Marciniak grooves on the local limit strains is illustrated in Fig. 6. As usual, f denotes the ratio of the initial sheet thickness within and outside the groove. It can be seen that the aforementioned conclusion remains valid for f sufficiently close to unity, although the range of limit local strains outside the groove decreased rapidly with f. For larger imperfections this range shrinks to a single value since then the critical stage [eqn (14)] is not reached in the uniform part of the sheet up to the onset of localized necking within the groove.

4. SHEET OF FINITE THICKNESS

In this section the assumption of plane stress is dropped, and we consider the two-dimensional boundary value problem in the (x_1, x_3) plane, defined in Section 2. The problem is analogous to the standard problem of plane strain tension, with the difference that the prescribed strain rate D_2 in the direction normal to the (x_1, x_3) plane is now non-zero. As shown below, this difference is associated with a qualitatively distinct post-critical solution.

In the numerical calculations reported here, the finite strain version of the J_2 corner theory of Christoffersen and Hutchinson (1979) has been assumed, with the same material parameters as in the examples from Section 3, with the exception (having little influence on the results) that the elastic constants are now taken to have more realistic values (the ratio of Young's modulus to the initial yield stress being equal to 500, and Poisson's ratio 0.3). The finite element method has been employed by using essentially the same calculation technique as that applied by Petryk and Thermann (1992) to the problem of plane strain tension; the reader is referred to that paper and to the references quoted therein for the details omitted here.

The calculations were performed for an initially perfect rectangular sheet, of initial dimension ratio $2l_0/h_0 = 100$, subject to balanced biaxial stretching ($\rho = 1$). The restriction to deformation symmetric with respect to the mid-planes of the sheet was introduced so that only one quadrant of the sheet cross-section in the (x_1, x_3) plane was actually computed. The quadrant was divided into 600×3 quadrilateral elements, each consisting of four "crossed" triangles of constant strain.

The algorithm implemented in the computer program makes it possible to cross bifurcation points with automatic rejection of the (unstable) fundamental post-bifurcation branch (Petryk and Thermann, 1992). During the present calculations, only one bifurcation point was found, and the instant and mode of the bifurcation practically coincided with

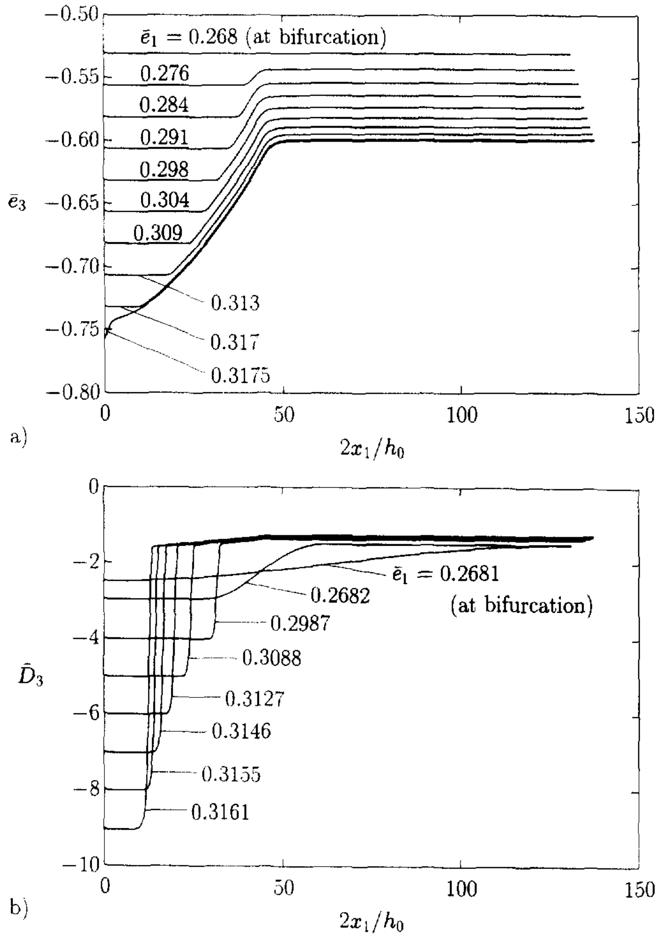


Fig. 7. Distribution of (a) thickness strain $\bar{e}_3$ and (b) thickness strain rate $\bar{D}_3$ in a sheet of initial ratio $2l/h_0 = 100$ at several stages of balanced biaxial stretching ($\rho = 1$, $N = 0.2$, $\theta_0 = 22.5^\circ$, $\theta_c = 135.0^\circ$).

those determined analytically by using the Hill and Hutchinson (1975) approach. The results presented below thus appear to be independent of the algorithm used to compute the post-bifurcation branch.

Although the strain-rate field just beyond the bifurcation point has the typical sinusoidal form, during subsequent deformation it undergoes rapid redistribution towards that determined in the preceding section under the plane stress assumption. After an increment of the overall logarithmic strain by 0.0001 from the bifurcation point, the two zones of practically constant strain rate are already developed. Figure 7 (a and b) show the distribution of the thickness strain $\bar{e}_3 = \ln(h/h_0)$ and its rate $\bar{D}_3$ at several stages of the post-critical deformation. It can be seen that the qualitative predictions of the plane stress theory from Section 3 concerning the evolution of the necking band are in full agreement with the calculations performed for a sheet of a finite thickness. The latter provide information, previously lacking, about deformations in the transitory zone.

The quantitative agreement is even more striking. This can be visualized by comparing the evolution of the width of the necking band (Fig. 8). The width b_h corresponding to the thickness distribution has been determined from eqn (19) with $h^{(b)}$ and $h^{(a)}$ interpreted as the minimum and maximum thickness, respectively. The width b corresponds now to the distribution of the rate of thickness strain and is defined by the formula analogous to eqn (15):†

† For an incompressible material and for the strain rate independent of x_3, the formulae (15) and (20) define the same quantity. For the slightly compressible material assumed, the difference exists but is not essential.

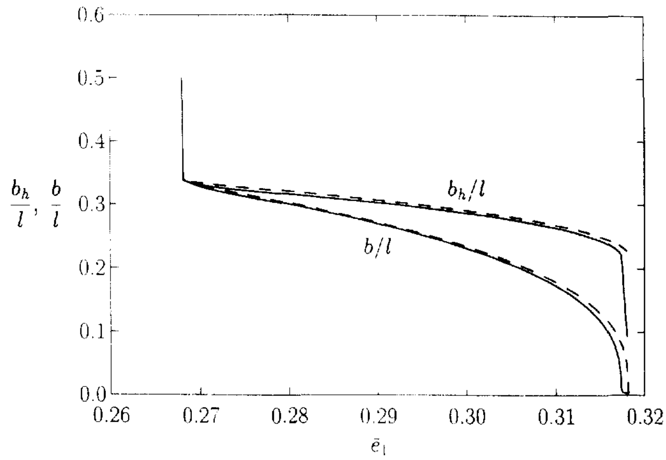


Fig. 8. Relative width of the current necking band, b/l, and of the neck, b_h/l, vs overall logarithmic strain $\bar{e}_1$. Solid lines for a finite sheet thickness correspond to the distributions shown in Fig. 7 (a, b) and dashed lines have been determined by plane stress calculations as in Section 3.

$$b = \frac{\bar{D}_\xi^{\text{mean}} - \bar{D}_\xi^{\text{max}}}{\bar{D}_\xi^{\text{min}} - \bar{D}_\xi^{\text{max}}} l. \tag{20}$$

Note the rapid drop of b and b_h from the initial value 0.5 l at bifurcation to the level predicted by plane stress calculations; the broken lines show the respective band width evolution determined as in Section 3 but for the changed elasticity constants. As expected, the curves start to diverge rapidly when the width of the necking band becomes of the order of the sheet thickness so that the assumption of plane stress is no longer acceptable.

With the exception of this last stage of the transitory process, the actual finite value of sheet thickness, provided small in comparison with l, has little influence on the numerical results. Just the fact that the sheet is treated as three-dimensional changes the mathematical character of the post-critical incremental problem since at least the *incipient* necking is not associated with the loss of ellipticity of the governing equations. The post-critical solution can thus be determined without the need for introducing any extra condition. It may be remarked, however, that the particular form of the solution illustrated in Fig. 7 is better explained by the plane stress theory for an infinitely thin sheet.

5. DISCUSSION AND CONCLUSIONS

The theoretical and numerical results presented above can be discussed from at least two different points of view: of a general theory of post-bifurcation behaviour in elastoplastic solids, and of the mechanics of sheet metal forming. Let us begin with the former.

The post-critical deformation of a biaxially stretched thin sheet of an incrementally non-linear material has been examined in two different ways. First, the plane stress idealization was assumed and the resulting indeterminacy of post-critical behaviour was removed by imposing an additional requirement of stability of the deformation process in the energy sense [eqn (11)]. Second, the assumption of plane stress was relaxed by introducing a finite sheet thickness, and then the post-critical deformation process could be determined in a more straightforward manner without any stability considerations (but at the cost of much greater computational effort). The results of both approaches have turned out to be in excellent agreement with each other which might not be obvious in advance. The agreement may be treated as an argument supporting the previous justification of the energy criterion of stability of a deformation path (Petryk, 1991).

The obtained interpretation of the point of ellipticity loss can also be of interest for the theory of post-bifurcation behaviour. It has been found that for incrementally non-linear solids the loss of ellipticity of the linearized equations of continuing equilibrium should not be identified with the onset of fully localized deformation. It marks the onset of instability of uniform straining but the post-critical deformations can concentrate gradually as further external displacements are applied. The "quasi-stable" process of non-uniform deformation can continue until another critical point—the loss of stability of *equilibrium* or (not studied here) of surface or bulk material stability—is reached.

Before proceeding to the discussion of the mechanism of formation of a localized neck in a metal sheet, it should be pointed out that the influence of factors such as anisotropy, strain-rate sensitivity, possibility of fracture or of shear band instability has been disregarded. The material model used in calculations is not fully satisfactory, and the boundary conditions also do not correspond closely to the experimental techniques used so far. Therefore, the conclusions have to be treated with due caution.

In previous studies it was usually assumed that strain localization in a plastically stretched sheet starts and develops within a *narrow* band containing a *fixed* portion of the material. The present work shows that that assumption may be incorrect if we take into account the incremental non-linearity of the material in accord with micromechanical theories of plasticity of polycrystalline metals. The width of the zone where the strain is concentrating can be initially of the order of the in-plane dimension and can decrease gradually in time until a localized neck is eventually formed. Accordingly, the limit strain outside the final band of localized necking is in general place-dependent and is reached on a non-proportional route, even for an initially perfect sheet subject to proportional overall stretching. The conclusion remains valid for a sheet with a local imperfection provided the imperfection is sufficiently small. The actual width of a necking band at early stages of the neck formation need not be determined by a small initial imperfection or by the sheet thickness. Rather, we have found that the band width evolution in an initially perfect sheet results from interaction of various material parameters, including those characterizing its incremental non-linearity associated with a vertex on the yield surface. From the point of view of predicting the sheet behaviour this is rather unfortunate since such material parameters are difficult to measure experimentally. On the other hand, a possibility is offered to determine such parameters indirectly, by precise measurements of the post-critical behaviour of the sheet. For instance, the formula (16) might be used to determine θ_0.

In view of the sensitivity of the post-critical behaviour to material parameters, boundary conditions and sheet imperfections, it is at present difficult to draw any definite conclusions from comparison with the available experimental data [cf. e.g. Marciniak and Kuczyński (1967); Azrin and Backofen (1970); Painter and Pearce (1974); Ghosh, (1978)], but at least no contradiction has been found so far. The present approach may be treated as an extension of the Stören–Rice (1975) theory, obtained by incorporating the analysis of post-critical behaviour, which leads to the unchanged predictions if $\bar{\bar{\phi}} < \theta_0$. The relative delay of the onset of localized necking predicted for $\phi > \theta_0$ (cf. Fig. 5) gives a possibility of obtaining a better agreement with experimental forming limit diagrams in the cases where the Stören–Rice curve is found to be insufficiently steep.

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APPENDIX A

Consider the problem of in-plane biaxial stretching of a sheet under the plane stress assumption at some stage of uniform deformation when the sheet is homogeneous and orthotropic with respect to (x_1, x_2) axes. We assume that the sheet currently occupies a rectangular domain G in the (x_1, x_2) plane as shown in Fig. 1, and that along the boundary ∂G of G the in-plane normal velocity and zero shear traction rate are prescribed.

Coefficients of the equations of continuing in-plane equilibrium expressed in terms of in-plane velocities $v_1(x_1, x_2)$, $v_2(x_1, x_2)$ under plane stress conditions [cf. eqn (8)] are denoted by $\hat{C}_{ijkl}$, $i, j, k, l = 1, 2$. The plane stress moduli $\hat{C}_{ijkl}$ are said to be strongly elliptic if

$$\hat{C}_{ijkl}a_i a_k b_j b_l > 0 \quad \text{for every non-zero } \mathbf{a}, \mathbf{b} \in \mathbf{R}^2. \tag{A1}$$

The following modified version of van Hove's theorem (van Hove, 1947) can be proved (Ryzhak, 1991).† If constant moduli $\hat{C}_{ijkl}$ are strongly elliptic and orthotropic with respect to the axes (x_1, x_2) then

$$\int_G \hat{C}_{ijkl} w_{i,j} w_{k,l} \, dA > 0 \tag{A2}$$

for every non-zero continuous field $\mathbf{w} : \bar{G} \to \mathbf{R}^2$ of a square integrable gradient and of zero normal component on ∂G. Here, dA is an infinitesimal area element in the (x_1, x_2) plane, and $(\cdot)_{,i}$ denotes the partial derivative with respect to x_i, with the summation convention for repeated indices. By the well-known argument [cf. Hill (1978)], eqn (A2) with constant $\hat{C}_{ijkl}$ ensures uniqueness of a solution to the respective linear boundary value problem for velocities.

The actual problem for velocities is non-linear since the moduli $\hat{C}_{ijkl}$ obtained from eqn (4) depend on the strain-rate direction. Uniqueness can still be concluded provided we have additional information about the material model, typically in the form of a constitutive inequality. This can be the relative convexity property (Hill, 1959, 1978), or the less restrictive inequality derived from micromechanical considerations (Petryk, 1989). For the version of the J_2 corner theory of plasticity (Christoffersen and Hutchinson, 1979) applied in this paper, it can be shown (Petryk, 1989, Section 8) that the latter inequality if satisfied along the fundamental path of uniform proportional stretching. The associated tangent moduli $\hat{C}^0_{ijkl}$ from the total loading cone are determined as in the deformation theory of plasticity and are orthotropic with respect to the axes (x_1, x_2). By combining the above statements, we arrive at the conclusion that during proportional stretching of a homogeneous and initially isotropic sheet, the uniform solution to the boundary value problem under consideration is unique so long as $\hat{C}^0_{ijkl}$ are strongly elliptic.

Numerical analysis has revealed (Stören and Rice, 1975) that for $0 \leqslant \rho \leqslant 1$ the moduli $\hat{C}_{ijkl}$ of the J_2 deformation theory cease to be strongly elliptic when $\hat{C}_{1111} = 0$, i.e. when the critical stage [eqn (14)] is reached. If $\rho < 1$ then at the critical stage we still have eqn (A1) when $a_2 \neq 0$ or $b_2 \neq 0$, and eqn (A1) becomes an equality for $a_2 = b_2 = 0$. Re-examination of the proof shows that eqn (A2) is still valid at this instant with the exception that the equality is obtained when $w_{1,1}$ is the only non-zero component of $w_{i,j}$, $i, j = 1, 2$. Hence, any bifurcation mode at the critical stage [eqn (14)] must be of that form and is thus independent of x_2; this motivated the selection of the problem for study as formulated in Section 2.

To be consistent with the micromechanical analysis, the total loading cone or its boundary should contain the current strain rate at every *regular* point on a non-proportional straining path (Petryk, 1989, p. 279). This is not so for the J_2 corner theory of plasticity used here, which may be regarded as an unsatisfactory feature of the

† Another and quite different proof of the modified theorem is omitted here

model. This may not be essential in calculations of smooth deformation paths but becomes more important when multiple bifurcations occur as in the plane stress example examined in the present work.

APPENDIX B

We provide below some details of the analysis from Section 3 along with the proofs of the statements formulated there.

For eqn (12) to hold, the inequality has to be satisfied in particular for $D_1 \to \pm\infty$ and for $D_1 \to D_1^0$. On substituting $\hat{U} = \frac{1}{2}\hat{C}_{jk}D_jD_k$ and using the continuity and homogeneity of the constitutive law, we obtain

$$\hat{C}_{11}(\pm 1, 0) \geqslant 0 \quad \text{at any } x_1 \in (0, l) \tag{B1}$$

$$\hat{C}_{11}(D_1^0, \bar{D}_2) \geqslant 0 \quad \text{at any } x_1 \in (0, l) \tag{B2}$$

as prerequisites for eqn (12), provided, of course, that the modulus $\hat{C}_{11}$ is well defined at these arguments.

The inequalities in eqns (12), (B1) and (B2) can be regarded as one-dimensional plane stress specifications for the sheet material, of the conditions of material stability discussed by Petryk (1992). Accordingly, the condition (B1) is interpreted as necessary for (dynamic) stability of equilibrium, at any boundary conditions. If the angle $\bar{\phi}$ between the plane strain rate $\mathbf{D}$ involved in eqn (B1) and $\bar{\mathbf{D}}$ is smaller than θ_0 then failure of eqn (B2) implies failure of eqn (B1). In typical circumstances where $\hat{C}_{11}$ is decreasing in time at any value of $D_1 \geqslant \bar{D}_1$, this gives no possibility of satisfying (B1) just beyond $\mathscr{H}^{\text{crit}}$. This justifies the statement about instability of post-critical equilibrium states in the case $\bar{\phi} < \theta_0$. For example, this is so for the classical elasto-plastic models for any value of ρ. For the rigid plastic version of the J_2 corner theory (Christoffersen and Hutchinson, 1979) with $\theta_0 = 22.5°$, this is so for $\rho < 0.2395$ if $E_t/E_s = 0.2$ and for $\rho < 0.4070$ if $E_t/E_s = 0.5$. For negligible elastic compliancies, θ_0 is identified with an angle of the total loading cone in stress-rate space.

If $\bar{\phi} > \theta_0$ at $\mathscr{H}^{\text{crit}}$ then usually $\hat{C}_{11}(\pm 1, 0) > 0$ at and just beyond $\mathscr{H}^{\text{crit}}$. From $\hat{C}_{11}(\pm 1, 0) > 0$ and from continuity and homogeneity of $\hat{U}$ it follows that $\hat{U}(D_1, \bar{D}_2) \to +\infty$ as $|D_1| \to \infty$. On the other hand, $\hat{C}_{11}(\bar{D}_1, \bar{D}_2)$ usually becomes negative on the fundamental path just beyond $\mathscr{H}^{\text{crit}}$ so that the graph of $\hat{U}$ as a function of D_1 at fixed $\bar{D}_2 > 0$ becomes non-convex and takes the form shown qualitatively in Fig. 2(c), possibly with more than two inflection points.

Consider any post-critical solution path involving a homogeneous zone where the graph of $\hat{U}$ vs D_1 has the qualitative form shown in Fig. 2(b) or (c). In analogy with the critical instant corresponding to Fig. 2(b), a family of secondary solutions can be constructed which are piecewise uniform in that zone and are generated by a pair of strain rates such that the respective stress rates, i.e. the slopes of the $\hat{U}$-graph, are equal to each other.† The non-convexity of the $\hat{U}$-graph implies the existence of infinitely many such pairs with their members taken from vicinities of the tangent points of a supporting straight line [the dashed line in Fig. 2(c)]. This demonstrates the existence of infinitely many velocity solutions at every point on any path under consideration. Along the fundamental post-critical path without necking, the secondary solutions are, moreover, energetically preferable to the uniform mode, in the sense of violation of eqn (11). This is the interpretation of the path instability in the energy sense for the present problem [cf. Petryk (1991, 1992)]; such a path is regarded as unrealizable in a physical system. Similarly, quasi-static bifurcation within a single and vanishingly narrow band without affecting the deformation path elsewhere violates the stability requirement outside the band and is regarded as unrealizable.

To avoid such instability, we must exclude $(\hat{C}_{11}(D_1^0, \bar{D}_2))^{\cdot} < 0$ everywhere at the critical instant. However, we can expect $(\hat{C}_{11}(D_1, \bar{D}_2))^{\cdot} < 0$ for every D_1 satisfying $\phi < \theta_0$ provided θ_0 is not too large; for instance, this is ensured for the total loading moduli of J_2 deformation theory with a power hardening law when $\theta_0 \leqslant \min(\bar{\phi}, \pi/2 - \bar{\phi})$, as can be shown by algebraic manipulations of the formulae. If this is so then we must have $\phi = \theta_0$ at every point of differentiability of v_1^0, i.e. $D_1^0(x_1)$ is equal to either D_1^- or D_1^+.

To illustrate better the meaning of eqn (16), suppose that the material obeys the J_2 corner theory of plasticity, developed by Christoffersen and Hutchinson (1979). In the limit when elastic compliancies are neglected, the angle ϕ is determined by the formula (4.5) in the paper just cited which for $D_1 > 0$ can be reduced to

$$\tan\phi = (3E_s/E_t)^{1/2}\frac{|\rho D_1 - \bar{D}_2|}{(2+\rho)D_1 + (1+2\rho)\bar{D}_2}, \tag{B3}$$

where E_t and E_s are the tangent and secant modulus, respectively, $E_s > E_t > 0$. In particular, the angle $\bar{\phi}$ between $\mathbf{D}$ and $\bar{\mathbf{D}}$ is defined by

$$\tan\bar{\phi} = (3E_s/E_t)^{1/2}\frac{\rho}{2+\rho}. \tag{B4}$$

For a power hardening law with an exponent N we have $E_s/E_t = 1/N$.

On determining the values D_1^- and D_1^+ from eqn (B3) for $\phi = \theta_0$, substituting them into eqn (15) and using (B4), we arrive again at eqn (16).

The equations (17) and (18) governing the necking band evolution represent the classical Weierstrass–Erdmann corner conditions of the calculus of variations, specified for the minimization problem (11); note that such $D_1^{(a)}$ and $D_1^{(b)}$ correspond to the tangent points of the $\hat{U}$-graph to a common straight line [the dashed line in Fig. 2(c)]. The post-critical solution is acceptable if the approximate‡ consistency condition $\dot{b} < bD_1^{(b)}$ is satisfied and if eqn (12) is not violated in zones (a) and (b). The solution can be continued unless b decreases

† In the case corresponding to Fig. 2(c), the mean strain rate within that zone is not given but is to be found from the condition [eqn (10)] of continuing equilibrium.

‡ The actual consistency condition is that the nominal width of zone (b) is decreasing. However, this width cannot be determined from the equations above.

eventually to zero, which on account of eqn (15) is necessarily accompanied by $D_1^{(b)} \to \infty$, so that the transition to localized necking is obtained. From eqns (10) and (7) we find that in the limit we must have $\hat{C}_{11}(1, \rho^{(b)}, \mathscr{H}^{(b)}) = \hat{C}_{11}(D_1^{(b)}, \bar{D}_2, \mathscr{H}^{(b)}) \to 0$ as $\rho^{(b)} \equiv \bar{D}_2/D_1^{(b)} \to 0$, so that the transition to localized necking is accompanied by approaching the limit of stability of equilibrium.

The case when the nominal width of zone (a) is decreasing in time can be examined in an analogous manner.